cross-section of our fictitious curved column, the constant of the fictitious bending foundation is (see Fig. 1)

$$C_B = \frac{Et^3}{12(1-\nu^2)}\left(2\int_{b/4}^{b/2} f'''' \, d\bar{s}\right)\frac{1}{w_0}; \quad (\,') = \frac{d\,(\,)}{d\bar{s}}$$

$$= \frac{8}{6}\,\frac{Et^3}{(1-\nu^2)}\left(\frac{\pi}{b}\right)^3 \tag{14}$$

and since the wavelength of the isolated column is obviously (see Fig. 1)

$$\bar{b} = b/2 \tag{15}$$

we can replace

$$t^3/24(1-\nu^2) \rightarrow t(b/2)^5/1440\, r^2 \tag{16}$$

and

$$t/2r^2 \rightarrow 8t^3\pi^3/12(1-\nu^2)\,b^3$$

Inserting the following simple attenuated buckling mode

$$w = a^{-is}e(\cos is + \sin is); \quad s = \varphi r$$

in the corresponding differential equation, a simple Galerkin procedure leads to

$$P^c = \tfrac{1}{2}(0.65\, Et^2/r^2) \cong 0.325\, E(t/r)^2 \tag{17}$$

Comparing the critical value obtained above with some of the experimental results, we find they are in good agreement with those reported by Fung and Sechler.[17] The most interesting point is that this value lies much lower than the value obtained by Friedrichs $P^c_{\min} = 0.9\,E(t/r)^2$ who has considered a transition condition from the buckled to the unbuckled region of the spherical shell which is somewhat similar to that corresponding to Eq. (19).

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Technical Comments

Comment on "Optimum Low-Thrust Rendezvous"

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REFERENCE 1 contains two linearized solutions for optimum low-thrust orbital rendezvous with ideal power-limited rocket engines. The solution for circular orbits is equivalent to the one obtained by Gobetz[2] more than a decade ago. The solution for elliptical orbits is restricted to small changes in radius and true anomaly. It is thus limited to short duration maneuvers and is somewhat incompatible with the low-thrust assumption which implies long maneuver duration. The author also claims that his elliptical orbit solution is valid for small eccentricity and longer duration. However, it is probably less accurate for these cases than his circular orbit solution.

The apparent reason for the additional restrictive assumption in the elliptical case is the desire to reduce the problem to a constant coefficient linear system, which is integrable. If the problem is linearized around a target in an elliptic orbit, it requires integration of a time varying linear system. This is apparently why the author of Ref. 1 introduced his additional assumptions. However, by using eccentric anomaly as the independent variable, it becomes possible to integrate this time varying linear system in terms of elementary functions.[3] The resulting solution is valid for arbitrary eccentricity and duration within the linear approximation.

In addition, by using a particular set of orbital elements as variables, additional results have been obtained for the practically interesting case of long duration maneuvers. In this latter case the optimal thrust programs become orthogonal, so that the optimal thrust program for each element produces no

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change in the other elements. The optimal thrust program for changing n elements is the vector sum of the n programs for changing each element individually.

This important result has been utilized in several further papers. An analytical solution[4] for the long duration transfer between arbitrary coplanar or arbitrary coaxial ellipses was obtained by Krylov-Bogoliubov averaging.[5] This nonlinear solution contains some instructive examples of the occurrance of conjugate points on optimal trajectories.[4,6]

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Reply by Author to T.N. Edelbaum

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IN connection with T.N. Edelbaum's comment on my paper[1] the following points might clarify the matter.

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1) T.N. Edelbaum's assertion concerning the solution for circular orbits given in Ref. 1 is not justified. Reference 2 published simultaneously with Ref. 3 is cited in order to show that Gobetz's variational problem differs from the variational problem in Refs. 1 and 3. The solution given in Ref. 1 completes the solution given by the author in Ref. 3 and one can speak of its equivalence with a solution given in Ref. 2 only on the basis of a comparative numerical application.

2) In Ref. 1 it is shown that the solution for elliptical orbits is valid both in the junction phase and in the terminal phase for maneuver durations compatible with the low thrust assumption.

The restriction for small changes of radius and true anomaly does not make incompatible the maneuver duration with low-thrust assumption as long as the magnitude of acceleration due to thrust does not exceed the imposed limits (at present $10^{-6} - 10^{-3}$g). With these values of acceleration due to thrust, the long maneuver durations are obvious in the interplanetary transfer or in the rendezvous on remote orbits. This is not the case treated in Ref. 1. The author's assumptions have permitted the easy integration of the equations of extremals without introducing substantial errors.

The use of the eccentric anomaly as an independent variable, recommended by T.N. Edelbaum, does not lead easily to rigorous analytical solutions both for optimal thrust program and for optimal trajectory. This can be clearly observed in Ref. 4.

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